Recall : $\gamma : [a,b] \to M$ is a geodesic if $D_{\gamma'}\gamma' = 0$.

$(\Rightarrow)$ ① $|\gamma'| \equiv$ const.

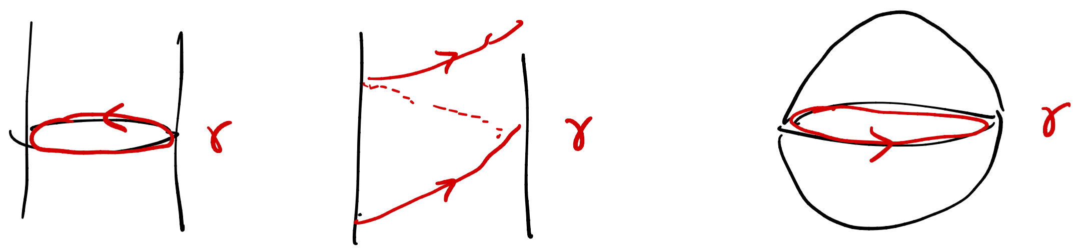
② $\gamma'' = k_n \cdot N + k_g \cdot n$ where $k_g \equiv 0$.

$(\Leftarrow)$ if $|\gamma'| = 1$ (WLOG) and $k_g \equiv 0$.

then $(\gamma'')^T = (k_n \cdot N + k_g \cdot n)^T = k_g \cdot n = 0$.

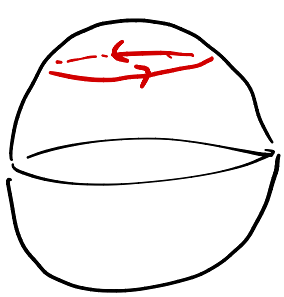
$\Rightarrow D_{\gamma'}\gamma' = 0$ #

Example : Suppose γ = parametrized by arc-length.



$\Rightarrow \gamma'' \perp T_\gamma M \quad \forall t. \Rightarrow k_g \equiv 0 \Rightarrow$ geodesic.

But



$\gamma'' \not\perp T_\gamma M$.

$\Rightarrow \gamma \neq$ geodesic.

given $\alpha : (a,b) \to M$ a regular curve.

$\alpha = X(u_1(t), u_2(t))$ s.t. $\alpha' = u_1' X_1 + u_2' X_2$.

$$\alpha'' = (u_i' X_i)'$$
$$= u_i'' X_i + u_i' (X_{ij} u_j')$$
$$= u_i'' X_i + u_i' (\Gamma_{ij}^k X_k u_j' + h_{ij} N u_j')$$
$$= (u_k'' + \Gamma_{ij}^k u_i' u_j') X_k + \underbrace{h_{ij} u_i' u_j'}_{= \mathrm{II}(\alpha', \alpha')} N.$$

$\therefore (D_{\alpha'}\alpha')^T \equiv 0$ iff $\boxed{u_k'' + \sum_{i,j} \Gamma_{ij}^k u_i' u_j' \equiv 0 \quad \forall k.}$ ☆☆ #

Corollary: If $\varphi: M \to N$ is isometry and $\alpha: (a,b) \to M$ is a geodesic, then $\beta = \varphi \circ \alpha : (a,b) \to N$ is also a geodesic.

pf: (Sketch)

Suffice to prove ☆☆ for β at each point.

Let $X: \mathcal{U} \to M$ be a parametrization of M at $p = \alpha(t_0)$

then $\tilde{X}: \mathcal{U} \to N$ given by $\tilde{X} = \varphi \circ X$ is a parametrization of N.

$$\begin{cases} \alpha' = u_1' X_1 + u_2' X_2 \\ \beta' = \tilde{u}_1' \tilde{X}_1 + \tilde{u}_2' \tilde{X}_2. \end{cases}$$

(thanks to the choice of parametrization)

$\Downarrow$

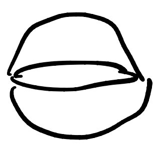
$\tilde{u}_1' = u_1'$, $u_2' = \tilde{u}_2'$.

$\because$ $\tilde{g}_{ij} = g_{ij}$ by isometry

$\therefore$ $\tilde{\Gamma}_{ij}^{k}(\varphi(p)) = \Gamma_{ij}^{k}(p)$, $\forall p \in \mathcal{U}$.

$\Rightarrow$ β satisfies geodesic equ. #

Example: $X(u,v) = \big(f(v)\cos u,\ f(v)\sin u,\ g(v)\big)$, $f>0$.

eg:  $\begin{cases} f(v) = \sin v \\ g(v) = \cos v. \end{cases}$ $\Rightarrow$ great circle = geodesic.

where great circle: $\alpha(t) = X(c_0, t)$.

$$\begin{cases} X_u = (-f\sin u,\ f\cos u,\ 0) \\ X_v = (f_v\cos u,\ f_v \sin u,\ g_v) \end{cases}$$

if $\alpha(t) = X(u_0, t)$, then

$$\alpha'' = (f_{vv}\cos u,\ f_{vv}\sin u,\ g_{vv})$$

$$\begin{cases} \langle \alpha'', X_u\rangle = 0 \\ \langle \alpha'', X_v\rangle = f_{vv} f_v + g_v g_{vv} \end{cases}$$

$$= \tfrac{1}{2}\left(f_v^2 + g_v^2\right)_v$$

$\therefore$ if $f_v^2 + g_v^2 \equiv \text{const} \neq 0$ along $\alpha(t)$ $\Rightarrow$ geodesic #

then ① $|\alpha'| \equiv$ constant ② $k_g = 0$

General fact from ODE:

Thm Let $\mathcal{U}$ be open set in $\mathbb{R}^n$.

Suppose $F: \mathcal{U} \times (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ is a smooth map. Then $\forall p \in \mathcal{U}$, $\exists \delta \in (0, \varepsilon)$ s.t. the initial value problem

$$* \begin{cases} X'(t) = F(x(t), t) \\ x(0) = p \end{cases}$$

admits a solution on $(-\delta, \delta)$.

Moreover, the sol. is unique.

$\Downarrow$ application.

prop: $\forall p \in M$, $\forall v \in T_pM$, $\exists \alpha : (-\varepsilon, \varepsilon) \to M$ which is a geodesic on $(-\varepsilon, \varepsilon)$ s.t.

$$\alpha(0) = p \quad , \quad \alpha'(0) = v.$$

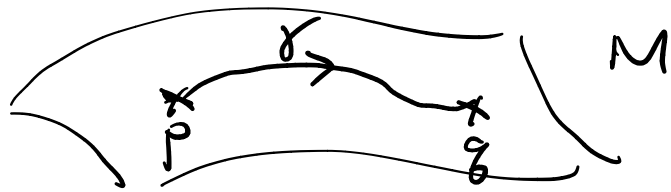
pf (Sketch): geodesic eqn $\Leftrightarrow$ $u_k'' + \Gamma_{ij}^k u_i' u_j' = 0$

Consider

$$* \begin{cases} \dfrac{dx_k}{dt} = y_k \\ \dfrac{dy_k}{dt} = -\displaystyle\sum_{i,j} \Gamma_{ij}^k y_i y_j \end{cases}$$

$\Rightarrow$ sol to $*$ solves geod. equ. #

Variational perspective:



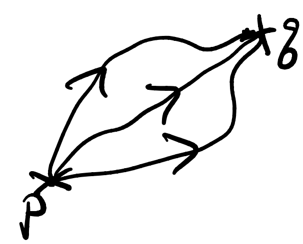
Let $\gamma : [a, b] \to M$ where $\begin{cases} \gamma(a) = p \\ \gamma(b) = q. \end{cases}$

length functional $L(\gamma) = \int_a^b |\gamma'| \, ds$.

Goal: minimize length functional.

Find local minimizer: $\gamma(t) = \alpha(t, s)\big|_{s=0}$

Let $\alpha : [a, b] \times (-\varepsilon, \varepsilon) \to M$ be a one-parameter family of curve, $\begin{cases} \alpha(t, 0) = \gamma(t) \\ \alpha(a, s) = p \\ \alpha(b, s) = q. \end{cases}$



Defn: $f(s) = L(\alpha(\cdot, s))$, then local minimal

$\Rightarrow f'(0) = 0$.

$$\frac{d}{ds}\Big|_{s=0} L(\alpha(s)) = \frac{d}{ds}\Big|_{s=0} \int_a^b \langle \alpha_t, \alpha_t \rangle^{\frac{1}{2}} \, dt$$

$$= \int_a^b \langle \alpha_t, \alpha_t \rangle^{-\frac{1}{2}} \langle \alpha_{ts}, \alpha_t \rangle \, dt \Big|_{s=0}$$

$$= \int_a^b \left\langle \frac{\partial}{\partial t}(\alpha_s), \frac{\alpha'}{|\alpha'|} \right\rangle dt \Big|_{s=0}$$

$$= -\int_a^b \left\langle \alpha_s, \frac{\partial}{\partial t}\left(\frac{\alpha'}{|\alpha'|}\right) \right\rangle dt \Big|_{s=0} \qquad \boxed{\because\ \alpha_s(a) = \alpha_s(b) = 0}$$

, $\forall\ W = \alpha_s$ where $W(a) = W(b) = 0$.

(i.e. $\forall$ variation fixing the end points.)

By taking $W = \phi \cdot \left(\frac{d}{dt}\left(|\gamma'|^{-1} \frac{d\gamma}{dt}\right)\right)^T$ where

$\phi = \overset{\geq 0}{\text{smooth}}$ fcn vanishing nearby boundary.

$$\Rightarrow \boxed{\therefore\ f'(0) = 0\ \ \forall W, \text{ iff } \left(\frac{d}{dt}\left(|\gamma'|^{-1}\frac{d\gamma}{dt}\right)\right)^T = 0.}$$

(OR proportional to)

$\therefore$ If $\gamma : [a,b] \to M$ is parametrised by arc-length and is critical point of length functional, then γ = geodesic.

Alternatively, define energy of γ by

$$E(\gamma) = \int_a^b \underbrace{|\gamma'|^2}_{\text{energy density (in } L^2)} dt$$

then for variation $\alpha : [a,b] \times (-\varepsilon, \varepsilon) \to M$ of $\gamma : [a,b] \to M$.

$$\frac{d}{ds}\Big|_{s=0} E(\alpha) = \frac{d}{ds}\Big|_{s=0} \int_a^b \langle \alpha', \alpha' \rangle \, dt$$

$$= \int_a^b \langle \partial_t \partial_s \alpha, \partial_t \alpha \rangle \, dt \Big|_{s=0}$$

$$\left(\begin{cases} \alpha(a,s) = p \\ \alpha(b,s) = q \end{cases} \right) \quad = -\int_a^b \langle \partial_s \alpha, \partial_t^2 \alpha \rangle \, dt \Big|_{s=0}$$

$\Rightarrow$ critical value iff $(\gamma'')^T = 0$.

Thm: Suppose γ = regular parametrised curve on $[a,b]$. γ = critical value of E iff γ = geodesic.

* wrt length, changing speed $\not\Rightarrow$ change in length

wrt energy, tangential acceleration $\Rightarrow$ Energy change.

*. $\text{length}(\gamma)^2 \leq (b-a) \cdot \text{Energy}(\gamma)$.

Equality holds iff arc-length parametrization

$\therefore$ minimization in length / Energy are geometrically equivalent.